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The Exterior Dirichlet Problem for a Quasilinear Elliptic Boundary Value Problem Suggested by Plane Shear Flow

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We study the existence of a classical solution of the exterior Dirichlet problem for a class of quasilinear elliptic boundary value problems that are suggested by plane shear flow. In this connection only bounded solutions which tend to zero at infinity are of interest. A priori bounds on solutions and constructive existence proofs are given. Finally, we prove the existence of a unique bounded solution of the shear flow and we show, under certain hypotheses about the asymptotic behavior of the nonlinearity, that this solution tends to zero at infinity. As an example, we consider the case of the parabolic shear flow.

1. PRELIMINARY HYPOTHESES AND DEFINITIONS

Let $\alpha \in (0, 1)$ be fixed. Denote by $D \subset \mathbb{R}^n$ $(n \geqslant 2)$ an unbonded domain having an internal closed boundary $\partial D \in C_{2+\alpha}$. We consider the second-order quasilinear differential operator

$$\mathfrak{L}u := \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} a_i(x, u)\, u_{x_i} + a_0(x, u)\, u, \tag{1}$$

with real coefficients $a_{ij} \in C_\alpha^{\mathrm{loc}}(D \cup \partial D)$ and a_i, $a_0 \in C_\alpha^{\mathrm{loc}}(D \cup \partial D \times \mathbb{R})$. $\mathfrak{L}$ is supposed to be uniformly elliptic, i.e., there exists a constant $\nu > 0$ such that, for all $x = (x_1, \ldots, x_n) \in D \cup \partial D$ and all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \geqslant \nu \sum_{i=1}^{n} \xi_i^2. \tag{2}$$

Let

$$f(x, z) \in C_{2+\alpha}^{\mathrm{loc}}(D \cup \partial D \times \mathbb{R}) \tag{3a}$$

and

$$g(x) \in C_{2+\alpha}(\partial D) \tag{3b}$$

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be given. Then we wish to find a bounded function $u(x)$ which belongs to $C_{2+\alpha}^{\mathrm{loc}}(D \cup \partial D)$ and satisfies

$$D: \quad \mathfrak{L}u = f(x, u) \tag{4a}$$

and

$$\partial D: \quad u = g. \tag{4b}$$

Let M be a positive constant. For any value $z \in [-M, M]$ we denote by

$$L_z v := \sum_{i,j=1}^{n} a_{ij}(x)\, v_{x_i x_j} + \sum_{i=1}^{n} a_i(x, z)\, v_{x_i} + a_0(x, z)\, v \tag{5}$$

the linear operator corresponding to $\mathfrak{L}$. Finally, let U_∞ be a neighborhood of infinity.

Definition 1. A function $v(x) \in C_2(U_\infty)$ is called a *barrier at infinity for the operator* $\mathfrak{L}$ (*or for the equation* $\mathfrak{L}u = 0$) if

(i) U_∞: $v > 0$,

(ii) $\lim_{|x| \to \infty} v(x) = 0$, $\quad |x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$,

(iii) $U_\infty \times [-M, M]$: $L_z v \leqslant 0$.

Definition 2. A pair of functions $(\underline{v}, \bar{v}) \in C_2(U_\infty) \times C_2(U_\infty)$ constitutes a *barrier at infinity for the equation* $\mathfrak{L}u = f(x, u)$ if

(i) U_∞: $\underline{v} < \bar{v}$,

(ii) $\lim_{|x| \to \infty} \underline{v}(x) = \lim_{|x| \to \infty} \bar{v}(x) = 0$,

(iii) $U_\infty \times [-M, M]$: $L_z \bar{v} \leqslant f(x, z) \leqslant L_z \underline{v}$.

Definition 3. A function $w(x) \in C_2(U_\infty)$ is called an *antibarrier at infinity for the operator* $\mathfrak{L}$ if

(i) $\lim_{|x| \to \infty} w(x) = \infty$,

(ii) $U_\infty \times [-M, M]$: $L_z w \leqslant 0$.

Let L be the linear uniformly elliptic second-order differential operator

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)\, u \tag{6}$$

with real coefficients a_{ij}, b_i, $c \in C_\alpha^{\mathrm{loc}}(D \cup \partial D)$.

We now consider the linear boundary value problem

$$\begin{aligned} D:&\quad Lu = F,\\ \partial D:&\quad u = g, \end{aligned} \tag{7}$$

where $F(x)$ and $g(x)$ are given continuous functions in $D \cup \partial D$, respectively, on ∂D. The above definitions are also valid for the equation, $Lu = F$ [4].

As in [4], we consider two different cases:

EXTERIOR DIRICHLET PROBLEM I_0. The solution $u(x)$ of (7) shall tend to zero as $|x|$ tends to infinity;

EXTERIOR DIRICHLET PROBLEM II. The solution $u(x)$ of (7) shall be bounded in D.

2. SOME AUXILIARY RESULTS

In this section we prove some theorems which follow from the Phragmèn–Lindelöf maximum principle. Some of these results are obtained in [3] under modified hypotheses.

THEOREM 1 (Phragmèn–Lindelöf [5]). *Let D be an unbounded (or bounded) domain, and let $u(x) \in C(D \cup \partial D) \cap C_2(D)$ satisfy*

$$\begin{aligned} D:&\quad Lu \geqslant 0\ [\leqslant 0],\\ \Gamma \subseteq \partial D:&\quad u \leqslant 0\ [\geqslant 0]. \end{aligned}$$

Suppose that:

(1) *There is an increasing sequence of bounded regions $D_1 \subset D_2 \subset \cdots \subset D_k \subset \cdots$ with the properties:*

(i) *each D_k is contained in D, such that for each $x \in D$ there is N_x such that $x \in D_n$ (and hence $x \in D_k$ for all $k \geqslant N_x$);*

(ii) *the boundary of each D_k consists of two parts $\Gamma_k \subseteq \Gamma \subseteq \partial D$, $\Gamma_k \neq \varnothing$, and $\Gamma_k' \subset D$.*

(2) *There is a sequence of functions $\{w_k\}$ with the properties:*

(i) $w_k\colon D_k \to R$,

(ii) $\bar{D}_k\colon w_k > 0$,

(iii) $D_k\colon Lw_k \leqslant 0$.

(3) *There is a function* $w(x)$ *with the property that at each point* $x \in D$ *the inequality* $w_k(x) \leqslant w(x)$ *holds for all* $k \geqslant N$.
If $u(x)$ *satisfies the growth condition*

$$\liminf_{k\to\infty} \left(\sup_{\Gamma_k'} \frac{u(x)}{w_k(x)} \right) \leqslant 0 \qquad \left[\limsup_{k\to\infty} \left(\inf_{\Gamma_k'} \frac{u(x)}{w_k(x)} \right) \geqslant 0 \right],$$

then $u \leqslant 0\ [\geqslant 0]$ *in* $D \cup \partial D$.

COROLLARY 1. *Let* D *be unbounded and* $c(x) \leqslant 0$ *in* $D \cup \partial D$. *Let* $u(x) \in C(D \cup \partial D) \cap C_2(D)$ *satisfy the inequalities*

$$D\colon \quad Lu \geqslant 0\ [\leqslant 0],$$
$$\partial D\colon \quad u \leqslant 0\ [\geqslant 0],$$
$$\limsup_{|x|\to\infty} u(x) \leqslant 0 \qquad [\liminf_{|x|\to\infty} u(x) \geqslant 0].$$

Then $u \leqslant 0\ [\geqslant 0]$ *in* $D \cup \partial D$.

Proof. The corollary follows from Theorem 1 with

$$D_k := \{x \mid x \in D, \mid x \mid < r_k\} \cap D, \qquad r_k \leqslant r_{k+1}, \tag{8}$$

$\Gamma = \partial D$, $w_k(x) = 1$, and $w(x) = 1$.

THEOREM 2 (see [3]). *Assume that*

(i) $D \cup \partial D$: $c \leqslant -c_0^2 = \text{const.} < 0$,

(ii) *the operator* L *has an antibarrier at infinity, which is defined and positive in* $D \cup \partial D$.

Then any bounded solution $u \in C(D \cup \partial D) \cap C_2(D)$ *of* (7) *satisfies*

$$| u(x)| \leqslant \max \left\{ \frac{\sup_{D\cup\partial D} | F(x)|}{\inf_{D\cup\partial D} | c(x)|}, \max_{\partial D} | g(x)| \right\} =: M \tag{9}$$

for all $x \in D \cup \partial D$.

Proof. Let $w(x)$ be the antibarrier at infinity of L, $w_k = w\,|_{D_k}$, with D_k given by (8). The functions $v_\pm(x) = M \pm u(x)$ satisfy

$$D\colon \quad Lv_\pm \leqslant 0,$$
$$\partial D\colon \quad v_\pm \geqslant 0,$$

According to Theorem 1 it follows that $v_\pm \geqslant 0$, or equivalently, $| u(x)| \leqslant M$ in $D \cup \partial D$.

COROLLARY 2. *Assume that L has an antibarrier at infinity, which is defined and positive in* $D \cup \partial D$. *Assume in addition that* $F \equiv 0$ *and* $c \leqslant 0$ *in* $D \cup \partial D$. *Then for any bounded solution* $u \in C(D \cup \partial D) \cap C_2(D)$ *of* (7) *we have for all* $x \in D \cup \partial D$ *the inequality*

$$|u(x)| \leqslant \max_{\partial D} |g(x)| . \tag{9'}$$

This corollary is proved in the same way as Theorem 2.

As an immediate consequence of the Phragmèn–Lindelöf principle we obtain the following uniqueness theorem.

COROLLARY 3 (see [3]). *The exterior boundary value problem* (7), *for which the operator L has an antibarrier* $w(x)$ *at infinity, with* $w > 0$ *in* $D \cup \partial D$, *cannot possess more than one bounded solution which belongs to* $C(D \cup \partial D) \cap C_2(D)$.

Proof. Let $w_k(x) = w(x)|_{D_k}$ with D_k given by (8). In order to prove this corollary, it is enough to apply Theorem 1 to the difference of two bounded solutions of the class $C(D \cup \partial D) \cap C_2(D)$.

THEOREM 3. *Assume that*

(i) $D \cup \partial D$: $c \leqslant -c_0^2 = \text{const.} < 0$,

(ii) *the equation* $Lu = F$ *has a barrier at infinity.*

Then the solution of the exterior Dirichlet problem I_0 *satisfies the inequality* (9).

Proof. According to [4] it follows that the exterior Dirichlet problem I_0 is well set for (7), i.e., there is a unique solution, $u(x) \in C_2(D)$, which tends to zero as $|x|$ tends to infinity. The functions $v_\pm(x) = M \pm u(x)$ with M given by (9) satisfy

$$\begin{aligned} D:&\quad Lv_\pm \leqslant 0,\\ \partial D:&\quad v_\pm \geqslant 0,\\ &\liminf_{|x|\to\infty} v_\pm \geqslant 0. \end{aligned}$$

By means of Corollary 1 we get $v_\pm(x) \geqslant 0$, or equivalently, $|u(x)| \leqslant M$ in $D \cup \partial D$.

3. EXISTENCE OF BOUNDED SOLUTIONS OF (4)

We prove the existence of bounded solutions of (4) by an explicit convergent iteration scheme. With some initial iterate $u^0(x)$ we define the sequence $\{u^n(x)\}$ by

$$\begin{aligned} D:\quad L_n u^n &:= \sum_{i,j=1}^{n} a_{ij}(x)\, u^n_{x_i x_j} + \sum_{i=1}^{n} a_i(x, u^{n-1})\, u^n_{x_i} + a_0(x, u^{n-1})\, u^n \\ &= f(x, u^{n-1}), \\ \partial D:\quad u^n &= g, \qquad \text{for } n = 1, 2, \ldots. \end{aligned} \tag{10}$$

In addition to assumptions (2) and (3) we require:

the functions a_{ij} are differentiable in D; (11a)

the functions a_{ij}, a_i, a_0, f and the partial derivatives of a_{ij} are bounded in $D \times \mathbb{R}$; (11b)

the operator $\mathfrak{L}$ has an antibarrier at infinity, which is defined and positive in $D \cup \partial D$; (12)

if $f \not\equiv 0$, then $|f| \leqslant m_1 = \text{const.}$ and $\alpha_0 \leqslant -m_0^2 = \text{const.} < 0$ in $D \times \mathbb{R}$. (13)

THEOREM 4. *Let* (2), (3), *and* (11)–(13) *be satisfied. Assume in addition that there is a pair of bounded functions* $(\underline{w}, \overline{w}) \in C_2(U_\infty) \times C_2(U_\infty)$ *such that* $L_z \overline{w} \leqslant f(x, z) \leqslant L_z \underline{w}$ *in* $U_\infty \times [-M, M]$ *with* M *given by* (9).

Then the boundary value problem (4) *possesses at least one bounded solution of class* $C^{\text{loc}}_{2+\alpha}(D \cup \partial D)$.

Proof. We define the first iteration $u^0(x)$ as the solution of the linear boundary value problem

$$\begin{aligned} D:\quad L_0 u^0 &:= \sum_{i,j=1}^{n} a_{ij}(x)\, u^0_{x_i x_j} + \sum_{i=1}^{n} a_i(x, 0)\, u^0_{x_i} + a_0(x, 0)\, u^0 \\ &= f(x, 0), \\ \partial D:\quad u^0 &= g. \end{aligned} \tag{14}$$

Using (2), (3), and the existence of pair $(\underline{w}, \overline{w})$ it follows from [4] that the exterior Dirichlet problem II for (14) is solvable in $C^{\text{loc}}_{2+\alpha}(D \cup \partial D)$. By means of (12), Corollary 3 implies the uniqueness of the solution $u^0(x)$. According to Theorem 2 we get

$$| u^0(x)| \leqslant \max \left\{ \frac{m_1}{m_0^2}, \max_{\partial D} | g(x)| \right\} = M \qquad \text{in} \qquad D \cup \partial D. \tag{15}$$

We now consider the linear boundary value problem (10). We can show in the same way as before, that, for each $n = 1, 2, \ldots$, $u^n(x) \in C^{\text{loc}}_{2+\alpha}(D \cup \partial D)$, u^n is unique, and $| u^n | \leqslant M$ in $D \cup \partial D$. According to [3, Theorem 1.3] the sequence $\{u^n\}$ is bounded in $C_\alpha(B)$, where B is a bounded subset of $D \cup \partial D$. Moreover,

$\| u^n \|_\alpha^B$ is independent of n. Hypotheses (1) and (2) allow the application of Schauder's interior estimates to each function u^n. It follows that the sequence $\{u^n\}$ is locally equicontinuous as well as locally bounded, whence one can extract a subsequence which converges uniformly on compact subsets of D to a solution u. That this solution takes on the boundary value $g(x)$ may be proved with the aid of Schauder's boundary estimates.

COROLLARY 4. *Let* (2), (3), (11), *and* (12) *be satisfied and suppose that* $a_0(x, z) \leqslant 0$ *in* $D \times \mathbb{R}$.

Then the boundary value problem (4) *with* $f \equiv 0$ *has at least one bounded solution which belongs to the class* $C_{2+\alpha}^{\mathrm{loc}}(D \cup \partial D)$.

The proof is almost identical to that of Theorem 4.

THEOREM 5. *Let* (2), (3), (11), *and* (13) *be satisfied and assume that the equation* $\mathfrak{L}u = f$ *has a barrier at infinity.*

Then the boundary value problem (4) *has at least one bounded solution of class* $C_{2+\alpha}^{\mathrm{loc}}(D \cup \partial D)$, *which tends to zero at infinity.*

Proof. Let $(\underline{v}, \bar{v}) \in C_2(U_\infty) \times C_2(U_\infty)$ be the barrier at infinity of the equation $\mathfrak{L}u = f$. Then, according to [4], it follows that the exterior Dirichlet problem I_0 is well set for (14) and also for (10), $n = 1, 2, \ldots$.

The fact that the sequence $\{u^n\}$ exists and converges uniformly on compact subsets of $D \cup \partial D$ to a solution u of (4) follows exactly as in Theorem 4. The only difficulty lies in showing that u tends to zero as $|x|$ tends to infinity.

Let $v_0(x)$ be a continuous function in $U_\infty := \{x \mid |x| > R\}$, so that

$$|x| \geqslant R: \quad \underline{v} \leqslant v_0 \leqslant \bar{v}. \tag{16}$$

By the methods of [4, Lemma 1] we can easily construct for each $n = 1, 2, \ldots$ a function w^n such that

$$\begin{aligned} |x| \geqslant R: &\quad \underline{v} \leqslant w^n \leqslant \bar{v}, \\ |x| > R: &\quad L_n w^n = f(x, u_{n-1}), \\ |x| = R: &\quad w^n = v_0 . \end{aligned} \tag{17}$$

Using (10) and (17) we obtain for $z^n(x) := u^n(x) - w^n(x)$

$$\begin{aligned} |x| > R: &\quad L_n z^n = 0, \\ |x| = R: &\quad -M - w^n \leqslant z^n \leqslant M - w^n, \\ &\quad \lim_{|x|\to\infty} z^n(x) = 0 \end{aligned}$$

with M given by (15). According to (16) and to the properties of the barrier we can find a constant $c > 0$ so large that

$$|x| = R: \quad -c(\bar{v} - \underline{v}) \leqslant -M - w^n \leqslant M - w^n \leqslant c(\bar{v} - \underline{v}).$$

By means of $L_n(\bar{v} - \underline{v}) \leqslant 0$ it follows that

$$|x| \geqslant R: \quad |z^n| \leqslant c(\bar{v} - \underline{v}),$$

or equivalently,

$$|x| \geqslant R: \quad -c\bar{v} + (c+1)\,\underline{v} \leqslant u^n \leqslant (c+1)\,\bar{v} - c\underline{v}$$

for each $n = 1, 2, \ldots$.

Since the right and left parts of the previous inequality are independent of n, we obtain for $n \to \infty$

$$|x| \geqslant R: \quad -c\bar{v} + (c+1)\,\underline{v} \leqslant u \leqslant (c+1)\,\bar{v} - c\underline{v},$$

whence

$$\lim_{|x| \to \infty} u(x) = 0,$$

Q.E.D.

If $f(x, u) \equiv 0$ then we have the following result, which is equivalent to Corollary 4.

COROLLARY 5. *Let* (2), (3), *and* (11) *be satisfied and suppose that* $a_0(x, z) \leqslant 0$ *in* $D \times \mathbb{R}$. *If the operator* $\mathfrak{L}$ *has a barrier at infinity, then the boundary value problem* (4), *with* $f \equiv 0$, *possesses at least one bounded solution of class* $C^{\mathrm{loc}}_{2+\alpha}(D \cup \partial D)$ *which tends to zero at infinity.*

4. PLANE SHEAR FLOW PROBLEM

We now consider the plane, incompressible, steady, and inviscid shear flow past a cylinder. Let D be the domain flow, i.e., the exterior of the section profile, $\partial D \in C_{2+\alpha}$, the boundary, and $U(x_2)$, the undisturbed velocity profile. The equation satisfied by the perturbed stream function $\psi_1(x_1, x_2)$ has the form

$$D: \quad \Delta\psi_1 = h(\psi_0 + \psi_1) - h(\psi_0), \tag{18}$$

with ψ_0 the unperturbed stream function,

$$\psi_0(x_2) = \int_0^{x_2} U(\xi)\, d\xi, \tag{19}$$

and $h(\cdot)$ a continuous function, which satisfies

$$h'(t) = \frac{U''(\psi_0^{-1}(t))}{U(\psi_0^{-1}(t))}. \tag{20}$$

Equation (18) can also be written as

$$D: \quad \Delta\psi_1 - a(x_2, \psi_1)\, \psi_1 = 0, \tag{21}$$

where

$$\begin{aligned} a(x_2, \psi_1) &:= \frac{h(\psi_0 + \psi_1) - h(\psi_0)}{\psi_1}, & \psi_1 \neq 0, \\ &:= h'(\psi_0), & \psi_1 = 0. \end{aligned}$$

The boundary conditions are

$$\partial D: \quad \psi_1 = -\psi_0, \tag{22}$$

$$\lim_{r\to\infty} \psi_1(x_1, x_2) = 0, \qquad r^2 = x_1{}^2 + x_2{}^2. \tag{23}$$

More details can be found in [1, 2, 6–9].

We apply some of our previous results to show that the problem (21)–(22) has a unique bounded solution. Moreover, under restrictive hypotheses about the asymptotic behavior of a we prove the existence of a unique solution of the problem (21)–(23).

THEOREM 6. *Assume that*

$$h(t) \in C_{1+\alpha}(\mathbb{R}), \tag{24}$$

$$\psi_0(x_2) \in C_{2+\alpha}(\partial D), \tag{25}$$

$$\mathbb{R}: \quad 0 \leqslant h' \leqslant c_1{}^2, \qquad c_1 > 0. \tag{26}$$

Then the shear flow problem (21)–(22) *has a unique bounded solution,* $\psi_1(x_1, x_2) \in C_{2+\alpha}^{\mathrm{loc}}(D \cup \partial D)$.

Proof. The function $w(x_1, x_2) = \ln(x_1{}^2 + x_2{}^2)^{1/2}$ is an antibarrier at infinity for Eq. (21). By means of Corollary 4, the existence of a bounded solution, $\psi_1(x_1, x_2) \in C_{2+\alpha}^{\mathrm{loc}}(D \cup \partial D)$ and $|\psi_1(x_1, x_2)| \leqslant \max_{\partial D} |\psi_0(x_2)|$ in $D \cup \partial D$ follows.

Further, let $\psi_1{}^*$ and ψ_1^{**} be two bounded solutions of (21)–(22) with $m^* \leqslant \tilde{\psi}_1 := \psi_1{}^* - \psi_1^{**} \leqslant M^*$. Set

$$D_1 := \{(x_1, x_2) \mid (x_1, x_2) \in D, \psi_1{}^*(x_1, x_2) > \psi_1^{**}(x_1, x_2)\},$$

and

$$D_2 := \{(x_1, x_2) \mid (x_1, x_2) \in D, \psi_1{}^*(x_1, x_2) < \psi_1^{**}(x_1, x_2)\}$$

and observe that D_1 and D_2 can be bounded or unbounded. For the sake of simplicity, and without loss of generality, we assume that D_1 is bounded and D_2 unbounded. Then, using (18) and (26), we obtain

$$\begin{aligned} D_1: &\quad \Delta\tilde{\psi}_1 = h(\psi_0 + \psi_1{}^*) - h(\psi_0 + \psi_1^{**}) \geqslant 0 \\ \partial D_1 \cap \partial D: &\quad \tilde{\psi}_1 = 0, \\ \partial D_1 \cap D: &\quad \tilde{\psi}_1 = 0. \end{aligned}$$

Therefore, by the maximum principle (applied to every component of D_1) [5] it follows that $\psi_1{}^* \leqslant \psi_1^{**}$ in $\bar{D}_1$, and hence, in $D \cup \partial D$. In $D_2 \cup \partial D_2$ we have

$$\begin{aligned} D_2: &\quad \Delta\tilde{\psi}_1 \leqslant 0, \\ \partial D_2 \cap \partial D: &\quad \tilde{\psi}_1 = 0, \\ \partial D_2 \cap D: &\quad \tilde{\psi}_1 = 0, \\ \limsup_{k\to\infty} \left(\inf_{\Gamma_k'} \frac{\tilde{\psi}_1}{w_k}\right) &\geqslant \limsup_{k\to\infty} \left(\inf_{\Gamma_k'} \frac{m^*}{w_k}\right) = 0, \end{aligned}$$

where $w_k(x_1, x_2) = w(x_1, x_2)|_{D_k}$ with D_k and Γ_k' defined as in Theorem 1. According to the Phragmèn–Lindelöf maximum principle (Theorem 1) it follows that $\psi_1{}^* \geqslant \psi_1^{**}$ in $D_2 \cup \partial D_2$, and hence, in $D \cup \partial D$. This means $\psi_1{}^* = \psi_1^{**}$ in $D \cup \partial D$. Q.E.D.

THEOREM 7. *Let* (24)–(26) *be satisfied and assume in addition*

$$U_\infty \times [-M, M]: \quad c_0{}^2/r^2 \leqslant a(x_2, z), \qquad c_0 > 0 \tag{27}$$

with $U_\infty := \{(x_1, x_2) \mid (x_1, x_2) \in D, r > R\}$ *a neighborhood of infinity and* $M := \max_{\partial D} \mid \psi_0(x_2)\mid$.

Then the shear flow problem (21)–(23) *has a unique solution* $\psi_1(x_1, x_2) \in C_{2+\alpha}^{\mathrm{loc}}(D \cup \partial D)$.

Proof. Theorem 6 gives the existence of a unique bounded solution $\psi_1(x_1, x_2) \in C_{2+\alpha}^{\mathrm{loc}}(D \cup \partial D)$ of the problem (21)–(22). It remains to show that this solution also fulfills condition (23). The function $v(r) = MR^{c_0}/r^{c_0}$, a solution of the boundary value problem

$$U_\infty: \quad \Delta v - \frac{c_0{}^2}{r^2} v = 0,$$

$$|x| = R: \qquad v = M > 0,$$

is a barrier at infinity for Eq. (21), because

$$U_\infty: \quad v > 0, \qquad \lim_{r\to\infty} v(r) = 0$$

and

$$L_z v = \Delta v - a(x_2, z)\, v \leqslant \Delta v - \frac{c_0^2}{r^2}\, v = 0.$$

The assertion follows then by Corollary 5.

As an example we consider the case of parabolic shear flow, i.e.,

$$U(x_2) = V_\infty(1 + cx_2^2) \tag{28}$$

with V_∞ and c two positive constants. For all $x_2 \in \mathbb{R}$,

$$0 \leqslant U''(x_2)/U(x_2) \leqslant 2c.$$

Since the range of the function a is included in that of the function h' and hence in that of $U''(x_2)/U(x_2)$ Eq. (26) follow. This implies the existence of a unique bounded solution of (21)–(22).

Moreover, condition (27) is also satisfied.

Indeed, from (28) we obtain

$$\begin{aligned}\psi_0^{-1}(t) &= (-qt + (q^2t^2 + p^3)^{1/2})^{1/3} - (qt + (q^2t^2 + p^3)^{1/2})^{1/3} \\ &= -2qt/\{p + (-qt + (q^2t^2 + p^3)^{1/2})^{2/3} + (qt + (q^2t^2 + p^3)^{1/2})^{2/3}\},\end{aligned}$$

with $q = -3/2V_\infty c$ and $p = 1/c$. For $t = \psi_0 + z$, with bounded z and large $|x_2|$, $\psi_0^{-1}(t)$ behaves as $O(|x_2|)$. Hence, for $|x_2| \geqslant x_2{}^*$ we have

$$2c/(1 + c^* x_2^2) \leqslant a(x_2, z).$$

On the other side, ψ_0^{-1} remains bounded for $|x_2| \leqslant x_2{}^*$. Therefore, for a convenient choice of the constant c_0, we obtain condition (27).

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